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SOME SUMS INVOLVING THE LARGEST AND SMALLEST PRIME DIVISOR OF A NATURAL NUMBER

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Some sums involving the Largest and Smallest Prime Divisor of a Natural Number

by

J. van de Lune

ABSTRACT

This report mainly deals with the asymptotic behaviour of some sums the terms of which being simple elementary functions of the largest and/or smallest prime divisor of a natural number.

KEY WORDS AND PHRASES: *Largest prime divisor, Smallest prime divisor*

For $n \in \mathbb{N}$, $n \geq 2$, let $g(n)$ (resp. $s(n)$) denote *the largest* (resp. *smallest*) *prime divisor* of n and let $g(1) = s(1) = 1$.

Throughout this report p and q will exclusively denote primes while empty sums (resp. products) have to be interpreted as 0 (resp. 1).

The main object of this report is to show that

(1) $$\sum_{n \leq x} \frac{g(n)}{s(n)} \sim c_1 \cdot \frac{x^2}{\log x}, \quad (x \to \infty)$$

where $c_1 = \frac{\pi^2}{12} \sum_p \{p^{-3} \prod_{q<p} (1-q^{-2})\}$.

(2) $$\sum_{n \leq x} \frac{1}{s(n)} \sim c_2 \cdot x, \quad (x \to \infty)$$

where $c_2 = \sum_p \{p^{-2} \prod_{q<p} (1-q^{-1})\}$.

(3.a) For every $A \in \mathbb{R}$

$$\sum_{n \leq x} \frac{1}{g(n)} = O_A(x (\log x)^{-A}), \quad (x \to \infty).$$

(3.b) There is <u>no</u> $\alpha < 1$ for which

$$\sum_{n \leq x} \frac{1}{g(n)} = O(x^{\alpha}), \quad (x \to \infty).$$

(4) $\sum_{n=1}^{\infty} \frac{1}{n^s g(n)}$ converges (absolutely) for $\mathrm{Re}(s) = \sigma \geq 1$ and diverges for $\sigma < 1$.

(5) $$\sum_{n \leq x} \frac{s(n)}{g(n)} = o(x), \quad (x \to \infty).$$

(6) $$\sum_{1<n\leq x} \frac{1}{\log g(n)} = o(x), \quad (x \to \infty).$$

PROOF of (1). Clearly, for any $N \in \mathbb{N}$, we have

$$\sum_{\substack{n \leq x \\ s(n)\leq N}} \frac{g(n)}{s(n)} \leq \sum_{n\leq x} \frac{g(n)}{s(n)} = \sum_{\substack{n \leq x \\ s(n)\leq N}} \frac{g(n)}{s(n)} + \sum_{\substack{n \leq x \\ s(n)>N}} \frac{g(n)}{s(n)} \leq$$

$$\leq \sum_{\substack{n \leq x \\ s(n)\leq N}} \frac{g(n)}{s(n)} + \frac{1}{N} \sum_{n\leq x} g(n).$$

In [1] BROUWER showed that

$$\sum_{n\leq x} g(n) \sim \frac{\pi^2}{12} \cdot \frac{x^2}{\log x}, \quad (x \to \infty).$$

Hence, using BROUWER's result, we obtain

$$\liminf_{x\to\infty} \frac{12}{\pi^2} \cdot \frac{\log x}{x^2} \cdot \sum_{n\leq x} \frac{g(n)}{s(n)} \geq \liminf_{x\to\infty} \frac{12}{\pi^2} \cdot \frac{\log x}{x^2} \cdot \sum_{\substack{n \leq x \\ s(n)\leq N}} \frac{g(n)}{s(n)}$$

and

$$\limsup_{x\to\infty} \frac{12}{\pi^2} \cdot \frac{\log x}{x^2} \cdot \sum_{n\leq x} \frac{g(n)}{s(n)} \leq \limsup_{x\to\infty} \frac{12}{\pi^2} \cdot \frac{\log x}{x^2} \cdot \sum_{\substack{n \leq x \\ s(n)\leq N}} \frac{g(n)}{s(n)} + \frac{1}{N}.$$

From this it is clear that, for our purpose, it is sufficient to determine the asymptotic behaviour of

$$\sum_{\substack{n \leq x \\ s(n)\leq N}} \frac{g(n)}{s(n)} = 1 + \sum_{p\leq N} \left\{\frac{1}{p} \sum_{\substack{n \leq x \\ s(n)=p}} g(n)\right\},$$

for some fixed $N \in \mathbb{N}$.

The important thing to do is to study the asymptotic behaviour of the sum

$$\sum_{\substack{n \leqq x \\ s(n)=p}} g(n),$$

where p is a fixed prime.

First let us consider the case p = 2. Then we have

$$\sum_{\substack{n \leqq x \\ s(n)=2}} g(n) = \sum_{2m \leqq x} g(2m) = 1 + \sum_{m \leqq \frac{x}{2}} g(m) \sim$$

$$\sim \frac{\pi^2}{12} \cdot \frac{(\frac{x}{2})^2}{\log \frac{x}{2}} \sim \frac{1}{2^2} \cdot \frac{\pi^2}{12} \frac{x^2}{\log x} .$$

Now let $p \geqq 3$. The natural numbers n with the property $s(n)=p$ can be described as $n = p(r+mp^*)$, $(m=0,1,2,3,\ldots)$, or equivalently by $\frac{n}{p} \equiv r(\bmod p^*)$, where $p^* = \prod_{q<p} q$, $(r,p^*) = 1$ and $0 < r < p^*$. Consequently we have

$$\sum_{\substack{n \leqq x \\ s(n)=p}} g(n) = \sum_{\substack{n \leqq x \\ \frac{n}{p} \equiv r(\bmod p^*) \\ (r,p^*)=1 \\ 0<r<p^*}} g(n) = p-1 + \sum_{\substack{m \leqq \frac{x}{p} \\ m \equiv r(\bmod p^*) \\ (r,p^*)=1 \\ 0<r<p^*}} g(m).$$

Thus, we have to investigate sums of the form

$$\sum_{\substack{n \leqq x \\ n \equiv a(\bmod b) \\ (a,b)=1 \\ 0<a<b}} g(n) = \sum_{\substack{0<a<b \\ (a,b)=1}} \sum_{\substack{n \leqq x \\ n \equiv a(\bmod b)}} g(n),$$

where $b \in \mathbb{N}$ is fixed (in our situation $b = p^* \geqq 2$). To this end we define

$$G_{a,b}(x) = \frac{12}{\pi^2} \frac{\log x}{x^2} \sum_{\substack{n \leqq x \\ n \equiv a(\bmod b)}} g(n), \quad (a \in \mathbb{Z};\ b \in \mathbb{N},\ x > 0)$$

and

$$S_b(x) = \frac{12}{\pi^2}\frac{\log x}{x^2} \sum_{\substack{n \leq x \\ n \equiv a (\mathrm{mod}\ b) \\ (a,b)=1 \\ 0 \leq a < b}} g(n), \quad (b \in \mathbb{N},\ x > 0).$$

REMARK. In case $b \geq 2$ the summation condition $0 \leq a < b$, in the definition of $S_b(x)$, may just as well be replaced by $0 < a < b$.
From these definitions it is clear that

$$S_b(x) = \sum_{\substack{0 \leq a < b \\ (a,b)=1}} G_{a,b}(x),$$

and

$$\sum_{0 \leq a < b} G_{a,b}(x) = \frac{12}{\pi^2}\frac{\log x}{x^2} \sum_{n \leq x} g(n),$$

so that

$$\lim_{x\to\infty} \sum_{0 \leq a < b} G_{a,b}(x) = 1.$$

We will write

$$G_{a,b} = \lim_{x\to\infty} G_{a,b}(x)$$

and

$$S_b = \lim_{x\to\infty} S_b(x)$$

whenever these limits exist.

Clearly, $G_{a,1}$ exists for all $a \in \mathbb{Z}$ and is equal to 1. Also, in case $G_{a,b}$ exists, we have $G_{a,b} = G_{c,b}$ whenever $a \equiv c \pmod b$. We will need the following

PROPOSITION. If $G_{a,b}$ exists then also $G_{ma,mb}$ exists for all $m \in \mathbb{N}$ and

$$G_{ma,mb} = \frac{1}{m^2} G_{a,b}.$$

PROOF.

$$G_{ma,mb}(x) = \frac{12}{\pi^2}\frac{\log x}{x^2}\sum_{\substack{n \leq x \\ n \equiv ma \pmod{mb}}} g(n) =$$

$$= \frac{12}{\pi^2}\frac{\log x}{x^2}\sum_{\substack{k \leq \frac{x}{m} \\ k \equiv a \pmod{b}}} g(mk) \leqq \frac{12}{\pi^2}\frac{\log x}{x^2}\sum_{\substack{k \leq \frac{x}{m} \\ k \equiv a \pmod{b}}} \{g(m) + g(k)\} \leqq$$

$$\leqq \frac{12}{\pi^2}\frac{\log x}{x^2}\left\{g(m)\cdot \frac{x}{m} + \sum_{\substack{k \leq \frac{x}{m} \\ k \equiv a \pmod{b}}} g(k)\right\} =$$

$$= o_m(1) + \frac{\log x}{x^2}\cdot\frac{(\frac{x}{m})^2}{\log\frac{x}{m}}\cdot G_{a,b}(\tfrac{x}{m}),\ (x > m),$$

and it follows that

$$\limsup_{x\to\infty} G_{ma,mb}(x) \leqq \frac{1}{m^2} G_{a,b}.$$

In a similar manner, using the obvious inequality $g(mk) \geqq g(k)$, one also proves that

$$\liminf_{x\to\infty} G_{ma,mb}(x) \geqq \frac{1}{m^2} G_{a,b}. \quad \square$$

Note that the above proof also yields:

$$G_{ma,mb}(x) = \frac{1}{m^2} G_{a,b}(\tfrac{x}{m}) + o_m(1),\ (x \to \infty).$$

We compute some $G_{a,b}$ and S_b. Since $G_{0,1} = 1$ we have $G_{0,b} = \frac{1}{b^2}$ for all $b \in \mathbb{N}$. In particular $G_{0,2} = \frac{1}{2^2}$, from which we obtain $G_{1,2} = (1-G_{0,2}=) 1 - \frac{1}{2^2}$. Hence $S_1 = 1$ and $S_2 = 1 - \frac{1}{2^2}$. Similarly we have

$$S_3 = \lim_{x\to\infty} S_3(x) = \lim_{x\to\infty}\{G_{1,3}(x) + G_{2,3}(x)\} =$$

$$= 1 - \lim_{x\to\infty} G_{0,3}(x) = 1 - G_{0,3} = 1 - \frac{1}{3^2}.$$

Some further examples are

$$S_4 = 1 - \frac{1}{2^2},\ S_5 = 1 - \frac{1}{5^2} \text{ and } S_6 = (1 - \frac{1}{2^2})(1 - \frac{1}{3^2}).$$

More generally we have

$$S_b(x) = \sum_{\substack{0 \leq a < b \\ (a,b)=1}} G_{a,b}(x) = \sum_{0 \leq a < b} G_{a,b}(x) - \sum_{\substack{0 \leq a < b \\ (a,b)>1}} G_{a,b}(x) =$$

$$= \sum_{0 \leq a < b} G_{a,b}(x) - \sum_{1<d|b} \sum_{\substack{0 \leq a < b \\ (a,b)=d}} G_{a,b}(x) =$$

$$= \sum_{0 \leq a < b} G_{a,b}(x) - \sum_{1<d|b} \sum_{\substack{0 \leq \frac{a}{d} < \frac{b}{d} \\ (\frac{a}{d},\frac{b}{d})=1}} G_{d\frac{a}{d}, d\frac{b}{d}}(x) =$$

$$= \sum_{0 \leq a < b} G_{a,b}(x) - \sum_{1<d|b} \sum_{\substack{0 \leq k < \frac{b}{d} \\ (k,\frac{b}{d})=1}} \left\{\frac{1}{d^2} G_{k,\frac{b}{d}}(\tfrac{x}{d}) + o_d(1)\right\}.$$

Using mathematical induction and taking limits $(x \to \infty)$ it follows that S_b exists for all $b \in \mathbb{N}$ such that

$$S_b = 1 - \sum_{1<d|b} \frac{1}{d^2} S_{\frac{b}{d}},$$

which may also be written as

$$\sum_{0<d|b} \frac{1}{d^2} S_{\frac{b}{d}} = 1.$$

Hence the convolution product (in the sense of DIRICHLET) of the arithmetical functions $\{\frac{1}{n^2}\}_{n=1}^{\infty}$ and $\{S_n\}_{n=1}^{\infty}$ is equal to E, where $E(n) = 1$ for all $n \in \mathbb{N}$. Since $\{\frac{1}{n^2}\}_{n=1}^{\infty}$ and E are multiplicative it follows immediately that $\{S_n\}_{n=1}^{\infty}$ is multiplicative: $S_{mn} = S_m \cdot S_n$ whenever $(m,n) = 1$.

In order to obtain an explicit formula for S_n we prove the following

LEMMA. *Let* α *be an arbitrary complex number and define the arithmetical functions* A *and* B *as follows:*

$$A(n) = n^{\alpha}$$

and

$$B(n) = \prod_{p|n} (1-p^{\alpha}).$$

Then $A * B = E$.

PROOF. The Dirichlet series associated with A and B are

$$\hat{A}(s) = \sum_{n=1}^{\infty} \frac{n^{\alpha}}{n^s} = \zeta(s-\alpha)$$

and

$$\hat{B}(s) = \sum_{n=1}^{\infty} \frac{B(n)}{n^s} = \text{(note that B is multiplicative)}$$

$$= \prod_p \left(1 + \frac{1-p^{\alpha}}{p^s} + \frac{1-p^{\alpha}}{p^{2s}} + \frac{1-p^{\alpha}}{p^{3s}} + \ldots\right) = \prod_p \left(1 + \frac{1-p^{\alpha}}{p^s-1}\right) = \frac{\zeta(s)}{\zeta(s-\alpha)} .$$

Hence $\hat{A}(s) \cdot \hat{B}(s) = \zeta(s)$, and the lemma follows.

Consequently, taking $\alpha = -2$, we obtain

$$S_n = \prod_{p|n} (1-p^{-2}).$$

Putting things together it is a simple exercise now to prove (1).

REMARKS. The existence of the numbers $G_{a,b}$ was proved only for the combinations a = 0, b = 1 (BROUWER's theorem) and a = 1, b = 2 and all cases derived from these by the formulas

$$G_{ma,mb} = \frac{1}{m^2} G_{a,b}$$

and

$$G_{a,b} = G_{c,b} \text{ if } a \equiv c (\text{mod } b).$$

The existence of the remaining $G_{a,b}$ was not relevant for our discussion, but this existence question seems to be interesting in itself. During the preparation of this report it was shown by M.R. BEST (oral communication) that the $G_{a,b}$ do exist indeed for all $a \in \mathbb{Z}$ and $b \in \mathbb{N}$. Moreover, he showed that $G_{a_1,b} = G_{a_2,b}$ whenever $(a_1,b) = (a_2,b) = 1$. From this it is clear that if $(d,n) = 1$ then

$$G_{d,n} = \frac{S_n}{\phi(n)} = \frac{1}{n} \prod_{p|n} (1 + \frac{1}{p}).$$

Hence, if $(d,ab) = (a,b) = 1$, then

$$G_{d,ab} = G_{d,a} \cdot G_{d,b}.$$

In view of the definition of the numbers $G_{a,b}$ this multiplicativity seems quite surprising.

By the method of proof for (1) we may also obtain the asymptotic behaviour of sums such as, for example,

$$\sum_{n \leq x} \frac{g(n)}{(s(n))^{\alpha}},$$

or more generally

$$\sum_{n \leq x} g(n) \cdot f(s(n)),$$

where f is some function satisfying certain conditions concerning its rate of growth for $n \to \infty$.

Under suitable circumstances we will have

$$\sum_{n \leq x} g(n) \cdot f(s(n)) \sim c_f \cdot \frac{x^2}{\log x}, \quad (x \to \infty),$$

where $$c_f = \frac{\pi^2}{12} \sum_p \{f(p)p^{-2} \prod_{q<p} (1-q^{-2})\}.$$

PROOF of (2). For any $N \in \mathbb{N}$ we have

$$\sum_{\substack{n \leq x \\ s(n) \leq N}} \frac{1}{s(n)} \leq \sum_{n \leq x} \frac{1}{s(n)} = \sum_{\substack{n \leq x \\ s(n) \leq N}} \frac{1}{s(n)} + \sum_{\substack{n \leq x \\ s(n) > N}} \frac{1}{s(n)} \leq$$

$$\leq \sum_{\substack{n \leq x \\ s(n) \leq N}} \frac{1}{s(n)} + \frac{1}{N} \sum_{\substack{n \leq x \\ s(n) > N}} 1 \leq \sum_{\substack{n \leq x \\ s(n) \leq N}} \frac{1}{s(n)} + \frac{x}{N}, \quad (x > 0).$$

Similarly as in the proof of (1) it suffices to determine the asymptotic behaviour of

$$\sum_{\substack{n \leq x \\ s(n) \leq N}} \frac{1}{s(n)} = 1 + \sum_{p \leq N} \sum_{\substack{n \leq x \\ s(n) = p}} \frac{1}{s(n)} = 1 + \sum_{p \leq N} \left\{\frac{1}{p} \sum_{\substack{n \leq x \\ s(n) = p}} 1\right\}.$$

Since the natural density of the positive integers n, having the property $s(n) = p$, is (see the description of these numbers on page 3)

$$\lim_{x \to \infty} \frac{1}{x} \sum_{\substack{n \leq x \\ s(n) = p}} 1 = \frac{1}{p} \prod_{q<p} (1 - \frac{1}{q}),$$

(2) follows easily. □

PROOF of (3.a). For positive x and y we define $\Psi(x,y) = \#\{n \in \mathbb{N} \mid n \leq x,\ g(n) \leq y\}$. Then we have (for $x > 1$)

$$\sum_{n\leq x} \frac{1}{g(n)} = \int_{1-0}^{x} \frac{1}{y}\, d\Psi(x,y) = \frac{1}{y}\Psi(x,y)\Big|_{y=1-0}^{y=x} - \int_{1}^{x} \Psi(x,y)dy^{-1} =$$

$$= \frac{1}{x}\Psi(x,x) + \int_{1}^{x} \frac{\Psi(x,y)}{y^2}\, dy.$$

Observe that $\Psi(x,x) = [x]$ for $x > 0$. In [2] DE BRUIJN showed that there are positive (absolute) constants K and c such that

$$\Psi(x,y) \leq K\, x \exp(-cu), \quad (x \geq 1;\ y \geq 2)$$

where $u = \dfrac{\log x}{\log y}$.

Hence for $x > 2$ we have

$$S(x) \overset{\text{def}}{=} \sum_{n\leq x} \frac{1}{g(n)} = \frac{[x]}{x} + \left(\int_{1}^{2} + \int_{2}^{x}\right) \frac{\Psi(x,y)}{y^2}\, dy \leq$$

$$\leq 1 + \Psi(x,2) + \int_{2}^{x} y^{-2} K \cdot x \cdot \exp\left(-c\,\frac{\log x}{\log y}\right) dy \leq$$

$$\leq O(\log x) + K \cdot x \cdot \int_{1}^{x} y^{-2} \exp\left(-c\,\frac{\log x}{\log y}\right) dy.$$

By the substitution $y = x^s$ the last integral is transformed into

$$(\log x) \cdot \int_{0}^{1} x^{-s} \exp\left(-\frac{c}{s}\right) ds = -\int_{0}^{1} \exp\left(-\frac{c}{s}\right) dx^{-s} = -\int_{0}^{1} \phi(s)\, dx^{-s},$$

where $\phi(s) = \exp(-\frac{c}{s})$, $0 < s \leq 1$ and $\phi(0) = 0$. Note that ϕ is infinitely many times differentiable on $[0,1]$ such that

$$\phi^{(n)}(0) = 0, \quad (n=0,1,2,3,\ldots).$$

Hence

$$-\int_0^1 \phi(s)dx^{-s} = -\phi(s)x^{-s}\Big|_{s=0}^{s=1} + \int_0^1 x^{-s}d\phi(s) =$$

$$= -\frac{\phi(1)}{x} - \frac{1}{\log x}\int_0^1 \phi'(s)dx^{-s} = -\frac{\phi(1)}{x} - \frac{\phi'(1)}{x\log x} + \frac{1}{\log x}\int_0^1 x^{-s}d\phi'(s) =$$

$$= \ldots\ldots = -\sum_{n=0}^{k} \frac{\phi^{(n)}(1)}{x(\log x)^n} + \frac{1}{(\log x)^k}\int_0^1 x^{-s}d\phi^{(k)}(s) =$$

$$= 0_k(\tfrac{1}{x}) + \frac{1}{(\log x)^k}\int_0^1 x^{-s}\phi^{(k+1)}(s)ds \leqq 0_k(\tfrac{1}{x}) + \frac{1}{(\log x)^k}\int_0^1 |\phi^{(k+1)}(s)|ds.$$

Hence

$$S(x) = 0(\log x) + 0_k(1) + 0_k\left(\frac{x}{(\log x)^k}\right) = 0_k\left(\frac{x}{(\log x)^k}\right),$$

proving (3.a). □

The proof of (3.b) will be given after (4) has been established.

PROOF of (4). Let $S(n) = \Sigma_{m=1}^{n} \frac{1}{g(m)}$. Then, using (3.a) with some $k > 1$, we have (from now on K will denote a positive constant not necessarily always the same) through summation by parts

$$\sum_{n=1}^{\infty} \frac{1}{ng(n)} = \sum_{n=1}^{\infty} (S(n)-S(n-1))\cdot\frac{1}{n} =$$

$$= \frac{1}{2} + \sum_{n=2}^{\infty} \frac{S(n)}{n(n+1)} \leqq \frac{1}{2} + K\cdot\sum_{n=2}^{\infty} \frac{1}{n^2}\cdot\frac{n}{(\log n)^k},$$

and it follows that the series under observation converges for $s = 1$ and hence (absolutely) for $\sigma \geqq 1$.

This result may also be proved as follows (not depending on DE BRUIJN's estimate for $\Psi(x,y)$):

$$\sum_{n=1}^{\infty} \frac{1}{ng(n)} = 1 + \sum_{p} \frac{1}{p} \Big(\sum_{g(n)=p} \frac{1}{n} \Big) =$$

$$= 1 + \sum_{p} \frac{1}{p} \Big(\frac{1}{p} \prod_{q \leqq p} (1 - \frac{1}{q})^{-1} \Big) \leqq 1 + K \cdot \sum_{p} \frac{\log p}{p^2}.$$

Here we used a weak version of MERTEN's theorem $\prod_{p \leqq x} (1 - \frac{1}{p}) \sim \frac{e^{-\gamma}}{\log x}$, $(x \to \infty)$. Now suppose that $\Sigma_{n=1}^{\infty} \frac{1}{n^s g(n)}$ converges for some s with Re(s) < 1. Then there is a non-negative $\sigma < 1$ for which $\Sigma_{n=1}^{\infty} \frac{1}{n^{\sigma} g(n)}$ converges. Hence

$$\sum_{n=1}^{\infty} \frac{1}{n^{\sigma} g(n)} = 1 + \sum_{p} \frac{1}{p} \Big(\sum_{g(n)=p} \frac{1}{n^{\sigma}} \Big) =$$

$$= 1 + \sum_{p} \frac{1}{p} \Big(\frac{1}{p^{\sigma}} \prod_{q \leqq p} (1 - \frac{1}{q^{\sigma}})^{-1} \Big) > \sum_{p} \frac{1}{p^{1+\sigma}} \prod_{q \leqq p} (1 - \frac{1}{q^{\sigma}})^{-1}.$$

Now we observe that

$$\log \prod_{q \leqq p} (1 - \frac{1}{q^{\sigma}})^{-1} = \sum_{q \leqq p} - \log(1 - \frac{1}{q^{\sigma}}) > \sum_{q \leqq p} \frac{1}{q^{\sigma}} = \int_1^p x^{-\sigma} d\pi(x) =$$

$$= \frac{\pi(p)}{p^{\sigma}} + \sigma \int_1^p \frac{\pi(x)}{x^{\sigma+1}} dx > \frac{\pi(p)}{p^{\sigma}} > K \frac{p}{p^{\sigma} \log p} = K \frac{p^{1-\sigma}}{\log p}.$$

Hence $$\prod_{q \leqq p} (1 - \frac{1}{q^{\sigma}})^{-1} > \exp(K \frac{p^{1-\sigma}}{\log p}) > K \frac{p^{k(1-\sigma)}}{(\log p)^k}, \quad (0 \leqq \sigma < 1).$$

We may choose k such that $k(1-\sigma) \geqq 1 + \sigma$. Then

$$\sum_{n=1}^{\infty} \frac{1}{n^{\sigma} g(n)} > K \cdot \sum_{p} \frac{p^{k(1-\sigma)-(1+\sigma)}}{(\log p)^k} = K \cdot \sum_{p} \frac{p^{\alpha}}{(\log p)^k}, \quad (\alpha > 0),$$

which clearly contradicts the assumption that $\Sigma_{n=1}^{\infty} \frac{1}{n^{\sigma} g(n)}$ converges. □

PROOF of (3.b). Suppose there exists an $\alpha < 1$ such that $S(x) = \sum_{n \leqq x} \frac{1}{g(n)} = O(x^{\alpha})$, $(x \to \infty)$. Clearly we may assume $\alpha > 0$. Choose σ such that $\alpha < \sigma < 1$.

Then we have

$$\sum_{n=1}^{\infty} \frac{1}{n^{\sigma} g(n)} = \sum_{n=1}^{\infty} (S(n)-S(n-1))\frac{1}{n^{\sigma}} =$$

$$= \sum_{n=1}^{\infty} S(n) \cdot \left\{\frac{1}{n^{\sigma}} - \frac{1}{(n+1)^{\sigma}}\right\} = \sum_{n=1}^{\infty} \frac{S(n)}{n^{\sigma}} \left(1 - (1+\frac{1}{n})^{-\sigma}\right) \leqq$$

$$\leqq K \sum_{n=1}^{\infty} \frac{n^{\alpha}}{n^{\sigma}} (1 - e^{-\frac{\sigma}{n}}) = K \sum_{n=1}^{\infty} \frac{1}{n^{1+\sigma-\alpha}} \cdot \frac{1-e^{-\frac{\sigma}{n}}}{\frac{\sigma}{n}} \leqq K \sum_{n=1}^{\infty} \frac{1}{n^{1+\sigma-\alpha}} .$$

Since $\sigma-\alpha > 0$ this contradicts (4), proving (3.b). Using DE BRUIJN's more refined estimates for $\Psi(x,y)$ in [2] it should be possible to obtain the "true" size of $S(x)$. □

PROOF of (5). First observe that the series

$$\sum_{p} \frac{1}{p} \prod_{q<p} (1 - \frac{1}{q})$$

converges (to the sum 1). This is a consequence of the identity

$$\sum_{k=1}^{n} a_k \prod_{r<k} (1-a_r) = 1 - \prod_{r=1}^{n} (1-a_r)$$

and the fact that

$$\lim_{x\to\infty} \prod_{q\leqq x} (1 - \frac{1}{q}) = 0.$$

Furthermore we have

$$\sum_{n\leqq x} \frac{s(n)}{g(n)} = \sum_{\substack{n \leqq x \\ s(n)\leqq N}} \frac{s(n)}{g(n)} + \sum_{\substack{n \leqq x \\ s(n)>N}} \frac{s(n)}{g(n)} ,$$

whereas

$$\frac{1}{x}\sum_{\substack{n\leq x\\ s(n)>N}}\frac{s(n)}{g(n)} \leq \frac{1}{x}\sum_{\substack{n\leq x\\ s(n)>N}} 1 = \frac{1}{x}\sum_{p>N}\sum_{\substack{n\leq x\\ s(n)=p}} 1 \leq$$

$$\leq \frac{1}{x}\sum_{p>N}\frac{x}{p}\prod_{q<p}\left(1-\frac{1}{p}\right) \stackrel{\mathrm{def}}{=} R(N) = o(1), \quad (N\to\infty).$$

Hence

$$\frac{1}{x}\sum_{n\leq x}\frac{s(n)}{g(n)} \leq \frac{N}{x}\sum_{n\leq x}\frac{1}{g(n)} + R(N)$$

so that

$$\limsup_{x\to\infty}\frac{1}{x}\sum_{n\leq x}\frac{s(n)}{g(n)} \leq R(N).$$

Since $R(N)\to 0$ when $N\to\infty$, we are done. □

PROOF of (6). Define $\lambda_n = \frac{\log n}{\log g(n)}$, $(n \geq 2)$ and let $\lambda_1 = 1$. Then

$$\sum_{1<n\leq x}\frac{1}{\log g(n)} = \sum_{1<n\leq x}\frac{\lambda_n}{\log n} = \sum_{\substack{1<n\leq x\\ n\leq N}}\frac{\lambda_n}{\log g(n)} +$$

$$+ \sum_{\substack{1<n\leq x\\ n>N}}\frac{\lambda_n}{\log g(n)} \leq K_N + \frac{1}{\log N}\sum_{n\leq x}\lambda_n.$$

In [5] VAN RONGEN showed that

$$\lim_{x\to\infty}\frac{1}{x}\sum_{n\leq x}\lambda_n = e^{\gamma},$$

where γ is Euler's constant. Hence

$$\limsup_{x\to\infty}\frac{1}{x}\sum_{n\leq x}\frac{1}{\log g(n)} \leq \frac{e^{\gamma}}{\log N},$$

and it follows that (6) holds. □

FINAL REMARK. The sum $\sum_{n\leq x} \log g(n)$ may be dealt with as follows. In [3] the author proved that

$$\lim_{x\to\infty} \frac{1}{x} \sum_{n\leq x} \frac{1}{\lambda_n} = a$$

with

$$a = -\int_1^\infty \frac{1}{x}\, dy(x)$$

where the continuous function $y:[0,\infty) \to \mathbb{R}$ is (uniquely) defined by

$$\begin{cases} y(x) = 1 & \text{for } 0 \leq x \leq 1 \\ y'(x) = -\frac{1}{x}\, y(x-1) & \text{for } x > 1. \end{cases}$$

Now observe that (writing $\sigma(x) = \sum_{n\leq x} \frac{1}{\lambda_n}$)

$$\sum_{n\leq x} \log g(n) = \sum_{n\leq x} \frac{1}{\lambda_n} \log n =$$

$$= \sum_{n\leq x} (\sigma(n) - \sigma(n-1))\log n = \text{(writing } k = [x])$$

$$= -\sigma(1)\log 2 + \sum_{n=1}^{k-1} \sigma(n)\log(1+\frac{1}{n}) + \sigma(k)\log k.$$

Hence, taking $x = m \in \mathbb{N}$, we obtain

$$\frac{1}{m}\Big(\sum_{n\leq m} \log g(n) - \sigma(m)\log m\Big) = -\frac{\sigma(1)\log 2}{m} +$$

$$+\frac{1}{m}\sum_{n=1}^{m-1} \frac{\sigma(n)}{n} \log(1+\frac{1}{n})^n = a + o(1), \quad (m\to\infty).$$

where a is defined as above. Hence $\sum_{n\leq m} \log g(n) = \sigma(m)\log m + a\cdot m + o(m)$.

Compare DE BRUIJN [2] who showed that $\sum_{n \leq x} \log g(n) = a\,x \log x + O(x)$. Finally we have for DICKMAN's sum

$$\sum_{1<n\leq x} \frac{\log g(n)}{\log n} = \sum_{1<n\leq x} \frac{1}{\lambda_n} \sim a \cdot x, \quad (x \to \infty).$$

Compare NORTON [4, pp.16-17]. It is also clear from [5] that

$$\sum_{1<n\leq x} \frac{\log n}{\log g(n)} = \sum_{1<n\leq x} \lambda_n \sim e^{\gamma} \cdot x, \quad (x \to \infty).$$

In [3] it was shown that $a > \gamma$. According to DICKMAN (cf [4], p.17) $a = 0.62433\ldots$ and one may check that

$$e^{\gamma} > a^{-1}, \quad (e^{\gamma} > 1.7 > \frac{5}{3} = \frac{1}{0.6} > \frac{1}{a}).$$

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